

Formulation of Solutions of a Class of Standard Cubic Congruence Modulo r^{th} Power of an Integer Multiple of n^{th} Power of Three

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Abstract: In this study, a class of standard cubic congruence modulo r^{th} power of a positive integer multiple of n^{th} power of three is formulated. The formula developed is tested and found true by solving different examples. First time a formula is developed for the solutions of the standard cubic congruence of the said type. Without formulation, it was very difficult for the readers to find the solutions of such cubic congruence. Formulation makes it possible to do so. This is the merit of the paper.

Keywords: Cubic congruence, Composite Number, Chinese Remainder Theorem, n^{th} -power modulus.

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Introduction

A congruence of the type: $x^3 \equiv a \pmod{p}$; p being a prime integer, is called a standard cubic congruence of prime modulus. It is said to be solvable if a is cubic residue of p [3]. Such congruence may or may not have solutions. The congruence $x^3 \equiv 3 \pmod{19}$ has no solutions whereas the congruence $x^3 \equiv 4 \pmod{13}$ has three solutions [1]. A congruence of the type: $x^3 \equiv a \pmod{m}$; m being a positive composite integer, is called a standard cubic congruence of composite modulus. It is found that some cubic congruence has exactly one solution such as: $x^3 \equiv 4 \pmod{6}$ has unique solution $x \equiv 4 \pmod{6}$; and some other has exactly three solutions such as $x^3 \equiv 8 \pmod{9}$ has three solutions $x \equiv 2, 5, 8 \pmod{9}$ [4]. Author formulated many standard quadratic congruence and also some standard cubic congruence of composite modulus. Here is the generalization of his previous paper on standard cubic congruence of composite modulus.

Literature Review

The standard cubic congruence has not been considered for study by earlier mathematicians and also not discussed systematically in the literature of mathematics. Much had been written on standard quadratic congruence. A detailed study of it is found in the literature of

mathematics. The author already formulated a class of standard cubic congruence of the type $x^3 \equiv b^3 \pmod{3^n}$ successfully with solutions: $x \equiv 3^{n-1}k + b \pmod{3^n}$ for $k = 0, 1, 2$, giving exactly three solutions [5].

Also, the cubic congruence of the type $x^3 \equiv a^3 \pmod{3^n \cdot b}$, for an integer $b \neq 3r$ with three solutions: $x \equiv 3^{n-1}bk + a \pmod{3^n \cdot b}$ for $k = 0, 1, 2$, [6].

Here the congruence under consideration is the generalisation of the above papers and have exactly three solutions.

Need of Research

The author found the standard cubic congruence a neglected chapter in Number Theory, and willingly go through the chapter and found much material for the research work. He (the author) tried his best to formulate the generalization of cubic congruence previously formulated and wish to present his effort in this paper. This is the need of this study.

Problem-Statement

Here, the problem is "To formulate the standard cubic congruence $x^3 \equiv a^3 \pmod{3^n \cdot b^r}$; b being a positive integer $b \neq 3l$; l, r, n are positive integers and to solve the examples using the formula established".

Analysis and Result

The congruence under consideration is: $x^3 \equiv a^3 \pmod{3^n \cdot b^r}$; $b \neq 3l$, is any positive integer. For its solutions, consider $x \equiv 3^{n-1} \cdot b^r k + a \pmod{3^n \cdot b^r}$; $k = 0, 1, 2, 3, 4, 5 \dots \dots$

$$\begin{aligned} \text{Then, } x^3 &\equiv (3^{n-1} \cdot b^r \cdot k + a)^3 \\ &\equiv (3^{n-1} \cdot b^r \cdot k)^3 + 3 \cdot (3^{n-1} \cdot b^r \cdot k)^2 \cdot a + 3 \cdot 3^{n-1} \cdot b^r k \cdot a^2 + a^3 \\ &\equiv a^3 + 3^n \cdot b^r (3^{2n-3} b^r k^3 + 3^{n-2} k^2 \cdot a + k a^2) \\ &\equiv a^3 \pmod{3^n \cdot b^r} \end{aligned}$$

Thus, it is a solution of the said congruence.

But for $k = 3$, it can be seen easily that $x \equiv a \pmod{3^n \cdot b^r}$ which gives the same result as for $k = 0$.

Similar results are for other higher values of k .

Therefore, it is concluded that $k = 0, 1, 2$ work and gives three solutions only.

These are: $x \equiv 3^{n-1} \cdot b^r k + a \pmod{3^n \cdot b^r}$ with $k = 0, 1, 2$.

Sometimes said congruence can be written as: $x^3 \equiv c \pmod{3^n \cdot b^r}$.

In this case, it can be written as: $x^3 \equiv c + k \cdot 3^n \cdot b^r \pmod{3^n \cdot b^r}$ [2]
 $\equiv a^3 \pmod{3^n \cdot b^r}$, if $c + 3^n \cdot b^r = a^3$

The solutions are given by as before.

Illustration

Consider the congruence: $x^3 \equiv 125 \pmod{5184}$. But $5184 = 64 \cdot 81 = 4^3 \cdot 3^4$.

Thus, it can also be written as: $x^3 \equiv 5^3 \pmod{4^3 \cdot 3^4}$.

It is of the type: $x^3 \equiv a^3 \pmod{3^n \cdot b^r}$ with $a = 5, n = 4, b = 4, r = 3$.

Such congruence has three solutions.

$$\begin{aligned} \text{The solutions are given by } x &\equiv 3^{n-1} \cdot b^r k + a \pmod{3^n \cdot b^r} \\ &\equiv 3^3 \cdot 4^3 k + 5 \pmod{3^4 \cdot 4^3} \end{aligned}$$

$$\begin{aligned} &\equiv 1728k + 5 \pmod{5184} \text{ with } k = 0, 1, 2. \\ &\equiv 0 + 5, 1728 + 5, 3456 + 5 \pmod{5184} \\ &\equiv 5, 1733, 3461 \pmod{5184}. \end{aligned}$$

Consider the congruence: $x^3 \equiv 64 \pmod{10125}$. But $10125 = 125 \cdot 81 = 3^4 \cdot 5^3$.
Thus, it can also be written as: $x^3 \equiv 4^3 \pmod{3^4 \cdot 5^3}$.

It is of the type: $x^3 \equiv a^3 \pmod{3^n \cdot b^r}$ with $a = 4, n = 4, b = 5, r = 3$.

Such congruence always has three solutions.

The solutions are given by $x \equiv 3^{n-1} \cdot b^r k + a \pmod{3^n \cdot b^r}$

$$\begin{aligned} &\equiv 3^3 \cdot 5^3 k + 4 \pmod{3^4 \cdot 5^3} \\ &\equiv 3375k + 4 \pmod{81 \cdot 125} \text{ with } k = 0, 1, 2. \\ &\equiv 0 + 4, 3375 + 4, 6750 + 4 \pmod{10125} \\ &\equiv 4, 3379, 6754 \pmod{10125}. \end{aligned}$$

Consider the congruence $x^3 \equiv 26 \pmod{486}$.

It can be written as $x^3 \equiv 26 + 486 = 512 = 8^3 \pmod{3^4 \cdot 6}$

It is of the type $x^3 \equiv a^3 \pmod{3^n \cdot b^r}$ with $a = 8, n = 4, b = 5, r = 1$.

It has three solutions given by $x \equiv 3^{n-1} \cdot b^r k + a \pmod{3^n \cdot b^r}$, $k = 0, 1, 2$.

$$\begin{aligned} &\equiv 3^3 \cdot 6k + 8 \pmod{3^4 \cdot 6} \\ &\equiv 162k + 8 \pmod{486} \\ &\equiv 8, 170, 332 \pmod{486}. \end{aligned}$$

Consider the congruence $x^3 \equiv 125 \pmod{3969}$.

Then the solutions are given by: $x \equiv 5, 1328, 2651 \pmod{3969}$ for $k = 0, 1, 2$.

Here lies the merit of formulation!!

Conclusion

Therefore, it can be concluded that the standard cubic congruence of the type $x^3 \equiv a^3 \pmod{3^n \cdot b^r}$; b being any positive integer, has exactly three solutions given by $x \equiv 3^{n-1} \cdot b^r k + a \pmod{3^n \cdot b^r}$ with $k = 0, 1, 2$.

Merit of the Paper

First time, a cubic congruence of the said type is considered for study and a formula is established to find all the solutions. Such type of standard cubic congruence is not yet formulated. Formulation is the merit of the paper.

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